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COMPSCI 225 Assignment 4

1.

a. We want to prove that, given a simple graph G with n vertices and a variable

k = (n choose 2), the number of graphs that can be made from G is 2^k.

First, observe that n choose 2 = n(n-1)/2 from the working shown below.

$$\binom{n}{2} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2!} = \boxed{\frac{n(n-1)}{2}}$$

Let's allow picking any pair of vertices act as drawing a line between them. In this context, k is then the number of possible edges in the graph because it is the number of ways we can pick 2 distinct vertices from graph G.

For each possible edge in our graph we have two choices: to either include it or not. Working our way around the graph, making this choice for each possible edge, we make k binary choices, leaving us with 2^k possible outcomes.

<u>Therefore, there are 2^k ways to draw the simple graph G.</u>

b. We want to show that, given a simple graph with n vertices, each having an even degree, and k = ((n-1) choose 2), there are 2^k different graphs that can be made.

We will start by leaving one vertex out, giving us n-1 remaining. Our solution to part (a) tells us that we can make 2^{n-1} graphs out of the remaining vertices.

When we add in our final point, all connections it makes in the will depend on whether the other vertex is even. We will connect it to every vertex of odd degree to make them even and not connect it any of even degree because that would make them odd.

Finally, the Handshaking Lemma tells us that the number of vertices with odd degree in any graph is even. This means that the number of vertices that the new point will connect to is also even, making its degree even as a result. There are 2^{n-1} ways to draw the initial graph, and only one way to add in the final point in order to satisfy our conditions. Therefore, there are 2^k graphs that can be made from n vertices with all vertices having an even degree.

2.

a. We want to show that given set A with m elements and set B with n elements, the number of 1 – 1 functions from A to B is n!/(n-m)!.

The formula given is the formula for the number of permutations of m elements from set of size n. We will use this as the basis of our proof.

The definition of 1 – 1 functions tells us that we must have m distinct elements in B for each element in A to map to its own.

Since all elements in A and B are distinct, we can then imagine lining up all inputs from A and assigning each a unique output in set B. This gives us n choices for the first, n-1 choices for the second, and so on. Here, we have clearly outlined a permutation problem.

<u>Because we have a permutation of m elements from a set of size n, we can say that the number of 1 – 1 functions from A to B is the permutation formula: n!/(n-m)!</u>

b. From part (a) we know that, given m = 5 and the number of possible functions = 6720, we can express the size of B as:

6720 = n!/(n-5)!, or 6720 = n(n-1)(n-2)(n-3)(n-4)

<u>Solving this equation for x (or a bit of trial and error) tells us that B but be of size 8 because 8 * 7 * 6 * 5 * 4 = 6720.</u>

3. We want to show that given a set (A) of 10 numbers from the range of 1 to 100, there exists two disjoint proper subsets (B and C) of A that have the same sum.

To start, we know that the number of possible subsets from a set of length 10 is 2^{10}, because for each element we have the choice to either include it in the subset or not. This comes out to 1024 total subsets, and 1023 proper subsets (excluding the set of all 10 numbers).

We will then look at the range of values we can achieve by adding together a proper subset of 10 numbers between 1 and 100. The empty set will have a sum of 0, and the set of 9 values of 100 will result in a sum of 900, giving 901 possible sums.

The pigeonhole principle tells us that with 1023 proper subsets and 901 possible values, there exists at least two distinct proper subsets with the same sum.

We are also told that B and C must be disjoint, which we can achieve by removing any shared elements. This will decrease the value of both sums by the same amount, guaranteeing that B and C still have the same sum.

<u>Therefore, for any set of 10 numbers from range 1 to 100, there exists two disjoint proper subsets with the same sum.</u>

4.

a. (Choose 1 denomination from 13) * (choose 4 cards from 4) *

   (choose 1 card from the 48 remaining)

   Solution shown below

b. (Choose 1 denomination from 13) * (choose 3 cards from 4) *

   (choose 2 denominations from the remaining 12) * (choose 1 card from 4) *

   (choose 1 card from 4)

   Solution shown below

4. a. $\binom{13}{1}\binom{4}{4}\binom{48}{1} = 13*1*48 = \underline{624}$

b. $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 13*4*66*4*4 = \underline{54,912}$

5.

a. Strings beginning with two zeros = 2^{25}

   Strings ending with three ones = 2^{24}

Strings beginning with two zeros and ending with three ones = 2^{22}

Strings either beginning with two zeros or ending with three ones =

$2^{25} + 2^{24} - 2^{22}$ = 46,137,344

b. We are essentially dividing 11 elements into 3 groups. This means we can choose 2 dividing points from a set of 10 gaps between elements. 10 choose 2 gives us 45 ways to divide up the elements, meaning there are 45 solutions to the equation given.

6. To find the number of 10 element subsets of the alphabet with a pair of consecutive letters, we first want to know how many subsets of size 10 have no consecutive letters, which we will use to calculate the result.

   Choosing 10 non-consecutive elements means we need 9 elements that work as "dividers". This means that if we reserve 9 total spaces for dividers, we can choose any 10 of the remaining 17 and insert the dividers afterwards to guarantee that no two elements are adjacent. It follows that the number of subsets with no consecutive letters is 17 choose 10.

   We will then take the total number of subsets of size 10, which is 26 choose 10, and find the difference.

   26 choose 10 = 5,311,735

   17 choose 10 = 19,448

   5,311,735 – 19,448 = 5,292,287 subsets of the alphabet with a pair of consecutive letters.