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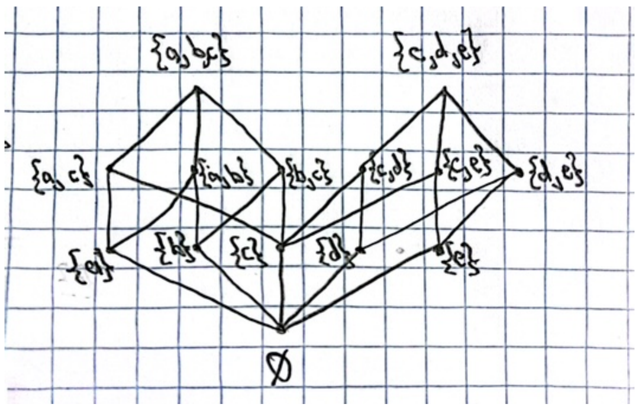
COMPSCI 225 Assignment 3

1.
   a. 

   b. Maximal elements: {a, b, c} and {c, d, e} Minimal elements: {}

2.
   a. We want to show that the n-cube graph Q_n for $n \geq 1$ is bipartite. We will do this by construction.
   An n-cube graph consists of vertices represented by a string of n bits, with edges between vertices that have only one bit flipped. In order for a graph to be bipartite, we must be able to partition it into two disjoint sets and show that all edges in the graph go between sets.
   Let A = the set of vertices with an even number of 1's
   Let B = the set of vertices with an odd number of 1's

   By definition, A and B are disjoint sets. We know that flipping only 1 bit will always change the number of 1's from even to odd or from odd to even. This tells us that an edge between two vertices will always go between set A and B.
   We have partitioned Q_n into two disjoint sets and shown that all edges in the graph go between the sets, and therefore we've shown that Q_n is bipartite.

   b. We want to prove that a simple graph is connected if it has n vertices and more than (n−1)(n−2)/2 edges. We will do this by direct proof.
   First, as has been proven in class, the maximum number of edges in a simple graph is n(n-1)/2 (making it a complete graph). In order for a graph to be disconnected it must have at least one vertex that exists as a separate component, leaving the main component with n-1 vertices. By plugging in n-1 to the formulation for the maximum number of edges in a simple graph, it follows that this main component will then have at most (n-1)((n-2)/2 edges.

We've outlined the situation with the maximum number of edges, given a simple graph is disconnected, which has led us back to the equation we started out with. Therefore, by our definition, if a simple graph has more than (n-1)(n-2)/2 edges then it must be connected.

c. They are isomorphic:
   f(a) = x
   f(b) = y
   f(c) = u
   f(d) = w
   f(e) = z
   f(f) = v

3.
a.
   i. 60 = 2 * 2 * 3 * 5 ➔ f(60)= 4
   ii. No. Take f(4) and f(6):
   4 = 2 * 2 ➔ f(4) = 2   6 = 2 * 3 ➔ f(6) = 2
   Since we have f(x) = f(y) and x != y, we know that f(n) is not injective.
   iii. To show that f(n) is surjective we need to show that for every positive integer (k) there is another positive integer (n) that has k prime divisors. Since we're counting multiplicities, we can use powers of two as an example. We can take 2^k as the n that will map to any positive integer k. Examples: 2^2 = 4, 2^3 = 8, 2^4 = 16, and so on. For any integer k, we can find an n such that f(n) = k, and therefore f(n) is surjective.

b.
   i. $f(-2) = (-1)^{-2} * 3 + (-2) = 1 * 3 – 2 = 1$ $f(5) = (-1)^5 * 3 + 5 = -1 * 3 + 5 = 2$
   ii. To prove f(n) has an inverse we need to prove it is bijective:
   <u>f(n) is injective:</u>
   $(-1)^a * 3 + a = (-1)^b * 3 + b$
   ➔ $3((-1)^a - (-1^b)) + a – b = 0$
   In order for this expression to be true, a and b must be the same number, therefore f(n) is injective.
   <u>f(n) is surjective:</u>
   If f(n) is odd then we can take the even number f(n) - 3 as n, which allows us to reach any odd number.
   If f(n) is even then we can take the odd number f(n) + 3 as n, which allows us to reach any odd number.
   We can reach any even integer and any odd integer, therefore n is surjective.

   f(n) is injective and surjective, meaning it is bijective and therefore invertible.

iii. $f(n) = 7$, which is odd and so by our rule from part (ii) we can choose $n = 7 - 3$
$\rightarrow f^{-1}(7) = 4$
To be sure, we will plug this into the function: $f(4) = (-1)^4 * 3 + 4 = 3 + 4 = 7$

4.

a. Equivalence classes of A:
{(1,3),(3,9)}
{(2,4),(−4,−8),(3,6)}
{(1,5)}

b. $a^3 = a^3 \pmod 7$ is true for any integer a, therefore R is reflexive
$a^3 = b^3 \pmod 7 \rightarrow b^3 = a^3 \pmod 7$ for all integers a and b, therefore R is symmetric
$a^3 = b^3 \pmod 7$ and $b^3 = c^3 \pmod 7 \rightarrow a^3 = c^3 \pmod 7$ for all integers a, b, and c, therefore R is transitive
Since R is reflexive, symmetric, and transitive, we know that it is an equivalence relation.
Equivalence classes of R:
$a^3 \pmod 7 = 0$
$a^3 \pmod 7 = 1$
$a^3 \pmod 7 = 6$

5.

a. We are given the function $f: A \rightarrow B$ such that $|A| = |B| = n$

i. We want to show that if f is injective, then f is surjective.
If f in injective then every element in A maps to a unique element in B. This means that there are n unique mappings of elements in A to elements in B. The definition of our function tells us that we have n elements in B, and since we have n mappings from A to B, every element in B must have a pre-image in A. This is the definition of a surjective function and therefore we know that if f is injective, then f is surjective.

ii. We want to show that if f is surjective, then f is injective.
If f in surjective then we know that every element in B has a pre-image in A. Since we are told that there are n elements in B, this means that there must be n pre-images in A. We also know that a function cannot map one input to two outputs, and so each pre-image must be is distinct. This tells us that there must be n elements in A to represent all n elements in B, and therefore each input must map to a distinct output. This is the definition of an injective function and therefore we know that if f is surjective, then f is injective.

b. The statement, “If $f : A \rightarrow B$ is a function which is one-to-one and $g : B \rightarrow C$ is onto, then $g \circ f$ is one-to-one” is false which we will prove by counterexample.
Take: A = {a, b}   B = {c, d}   C = (e}
We can define f as the injective function:
f(a) = c
f(b) = d
And define g as the surjective function:
f(c) = e
f(d) = e
Then observe $g \circ f$:
g(f(a)) = e
g(f(b)) = e
In this case, two different inputs map to the same output which means $g \circ f$ is not injective and therefore the statement is false.

6.

a. There is no simple graph with the degree sequence: [0, 2, 0, 3, 1, 0, 1, 1].
The sum of the numbers in the sequence gives the total number of vertices:
2 + 3 + 1 + 1 +1 = 8
Lets add in one vertex degree at a time:
We have one vertex of degree seven which means it must be connected to every other point. We also have one vertex of degree six which means it connects to every other point except one. This leaves us with only one vertex that connects to only one other point, with all others connecting to more than one. Already, we can see that this simple graph is not possible because we would need at least two vertices to be remaining of degree one or less.

b. Graph G with degree sequence [0, 25, 9, 0, x, y] has a total degree of:
25*1 + 9*2 + x*4 + y*5 = 43 + 4x + 5y
We are told that G has 30 edges which means it has a total degree sequence of:
2(30) = 60
It follows that:
60 = 43 + 4x + 5y ➔   17 = 4x + 5y
We know that x and y must both be integers and so the only combination of x and y for which 4x + 5y = 17 is x = 3 and y = 1.