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COMPSCI225 Assignment 2

1. We want to prove that gcd(3a+b, a) = gcd(a, b) for every two positive integers a, b in the set of natural numbers. We will do this by direct proof.

   First, we can say that gcd(x, y) = gcd(x+k*y, y) because gcd(k*y, y) = y
   Let: x = 3a + b, y = a, and k = -3
   We can then write: gcd(3a+b, a) = gcd((3a+b)+(-3*a), a) = gcd(b, a)
   The commutative property tells us that gcd(b, a) = gcd(a, b)
   And therefore, gcd(3a+b, a) = gcd(a, b)

   From this we can conclude that the proof holds.
2.
   a. We want to show that for every integer n, if n = 3 (mod 40), then n = 3 (mod 4) and n = 3 (mod 10). We will do this with proof by cases.

      <u>Case 1: If n = 3 (mod 40), then n = 3 (mod 4)</u>
      We can start by writing n as: n = 40k + 3 by the division theorem.
      We can then take n = (40k + 3) (mod 4).
      We know that 40k = 0 (mod 4) because 40 = 0 (mod 4), and so (40k + 3) = 3 (mod 4).
      Therefore, n = 3 (mod 4) and this case holds true.

      <u>Case 2: If n = 3 (mod 40), then n = 3 (mod 10)</u>
      Again, we can start by writing n as: n = 40k + 3 by the division theorem.
      We can then take n = (40k + 3) (mod 10)
      We know that 40k = 0 (mod 10) because 40 = 0 (mod 10), and so (40k + 3) = 3 (mod 10). Therefore, n = 3 (mod 10) and this case holds true.

      Since both cases hold true, we can say that for every integer n, if n = 3 (mod 40), then n = 3 (mod 4) and n = 3 (mod 10).

   b. We want to show that the statement "if n = 3 (mod 4) and n = 3 (mod 10) then n = 3 (mod 40)" is false. We will do this by counterexample.

      The least common multiple of 4 and 10 is 20. Since 4 and 10 both divide 20, we know that (20+3) = 3 (mod 4) and (20+3) = 3 (mod 10). Knowing this, set n = 23 and check if n = 3 (mod 40):
      23 ≠ 3 (mod 40) and therefore the statement is false.

3.

a. We want to prove that for every n ∈ N, n ≥ 1, we have the $\sum_{i=1}^{n} \mathrm{i}(3\mathrm{i}-1)$ = $n^2(n + 1)$. We will do this by induction.

Base case (n = 1): 1(3*1 – 1) = 1^2(1+1) ➔ 1(2) = 1(2) ➔ 2 = 2
The base case holds true.

Inductive case (P(n) ➔ P(n+1)):
To do this we want to express $\sum_{i=0}^{n+1} \mathrm{i}(3\mathrm{i}-1)$ as $(n+1)^2((n+1)+1)$
We will start with:
$\sum_{i=1}^{n+1} \mathrm{i}(3\mathrm{i}-1)$ = ($\sum_{i=1}^{n} \mathrm{i}(3\mathrm{i}-1)$) + (n+1)(3(n+1)-1)
= n^2(n+1) + (n+1)(3n+2)
= (n+1)(n^2+(3n+2))
= (n+1)(n^2+3n+2)
= (n+1)(n+1)(n+2)
= $(n+1)^2$(n+2)
Therefore, $\sum_{i=1}^{n+1} \mathrm{i}(3\mathrm{i}-1)$ = $(n+1)^2((n+1)+1)$ and the inductive case holds.

Since the base case and inductive case both hold true, the proof holds true.

b. We want to show that for every n ∈ N, n ≥ 1, we have $\sum_{i=1}^{n} 1/(2^\wedge \mathrm{i})$ < 1. We will do this by induction.

Base case (n = 1): 1/(2^1) < 1 ➔ ½ < 1
This case holds true.

Inductive case (P(n) ➔ P(n+1)):
We will start with:
$\sum_{i=1}^{n+1} 1/(2^\wedge(\mathrm{i}+1))$ = ($\sum_{i=1}^{n} 1/(2^{\mathrm{i}})$) + $1/2^{n+1}$
We know that $\sum_{i=1}^{n} 1/(2^\wedge \mathrm{i})$ can also be expressed as 1 - $1/2^n$).
Which mean that we can rewrite the equation for P(n+1) as:
1 - $1/2^n$ + $1/2^{n+1}$ = 1 – ($1/2^n$ - $1/2^{n+1}$)
We know that $1/2^n$ will always be larger than $1/2^{n+1}$ because 2 is larger than 1, in the denominator, and is being raised to a lower power in $1/2^n$ than $1/2^{n+1}$. Therefore, the term $1/2^n$ - $1/2^{n+1}$ will always be positive, leading to 1 – ($1/2^n$ - $1/2^{n+1}$) always being less than 1. From this we can conclude that the inductive case holds true.

Since the base case and inductive case both hold true, the proof holds true.

4. Given the game described, we want to show that the first player has a winning strategy <u>if and only if</u> n != 1 (mod 7). We will do this by strong induction, calling the first player "player A" and the second player "player B".
To prove that player A has a winning strategy <u>if and only if</u> n != 1 (mod 7) we must show that n != 1 (mod 7) ➔ player A has a winning strategy <u>AND</u> n = 1 (mod 7) ➔ player B has a winning strategy.

<u>Base cases (n = 1:14):</u>
Since we are using strong induction, we will look at the first 14 values of n as our base cases:
n = 1: player A takes 1 and loses
n = 2: player A takes 1, player B takes 1 and loses
n = 3: player A takes 2, player B takes 1 and loses
n = 4: player A takes 3, player B takes 1 and loses
n = 5: player A takes 4, player B takes 1 and loses
n = 6: player A takes 5, player B takes 1 and loses
n = 7: player A takes 6, player B takes 1 and loses
n = 8: player A takes i ∈(1, 6) player B takes 7-i, player A takes 1 and loses
n = 9: player A takes 1, player B takes i ∈(1, 6) player A takes 7-i, player B takes 1 and loses
n = 10: player A takes 2, player B takes i ∈(1, 6) player A takes 7-i, player B takes 1 and loses
n = 11: player A takes 3, player B takes i ∈(1, 6) player A takes 7-i, player B takes 1 and loses
n = 12: player A takes 4, player B takes i ∈(1, 6) player A takes 7-i, player B takes 1 and loses
n = 13: player A takes 5, player B takes i ∈(1, 6) player A takes 7-i, player B takes 1 and loses
n = 14: player A takes 6, player B takes i ∈(1, 6) player A takes 7-i, player B takes 1 and loses

We can see that when n != 1 (mod 7) player A has a winning strategy and when
n = 1 (mod 7) player B has a winning strategy, and therefore our base cases hold true.

<u>Inductive cases:</u>
As we see in the base cases, to be in the winning position player A must be able to leave player B with m = 1 (mod 7) coins. For any value where n != 1 (mod 7), player A can take j coins where j+1 = n (mod 7), j ∈(1, 6), which keeps player A in the winning position. Since player A can guarantee that they stay in the winning position given any starting value where n != 1 (mod 7), we can say that player A is guaranteed to have a winning strategy in this case.

Conversely, if player A starts with n = 1 (mod 7) coins then this means that player B starts in the winning position. Therefore, for any number, i, coins that player A takes, player B can take 7-i coins and keep themselves in the winning position. Since player B can guarantee that they stay in the winning position given any situation where n = 1 (mod 7), we can say that player B is guaranteed to have a winning strategy in this case.

Since n != 1 (mod 7) ➔ player A has a winning strategy <u>AND</u> n = 1 (mod 7) ➔ player B has a winning strategy, we can conclude that:
player A has a winning strategy <u>if and only if</u> n != 1 (mod 7) and the whole proof holds true.

5.

a. We want to show that for f : R → R is convex, n ∈ N, and n ≥ 2, the following holds:
For any $x_1, x_2, \ldots, x_n \in R$ and $p_1, \ldots, p_n \in (0, 1)$ such that $p_1 + \ldots + p_n = 1$, we have:

$$f(\sum_{i=1}^{n} pi * xi) <= \sum_{i=1}^{n} pi * f(xi)$$

We will do this using induction and the definition of a convex function.

<u>Base case (n = 2):</u>
Writing the summation when n = 2 explicitly gives us:

$$f(p_1x_1 + p_2x_2) <= p_1 f(x_1) + p_2f(x_2)$$

And because we know that $p_1 + p_2 = 1$, this is equal to the definition of a convex function. Because we know that the function is convex, we can say this expression is true and therefore our base case holds true.

<u>Inductive case (P(n) ➔ P(n+1)):</u>
We want to show that: $f(\sum_{i=1}^{n+1} pi * xi) <= \sum_{i=1}^{n+1} pi * f(xi)$

which can be written as:
$f(\sum_{i=1}^{n-1} pi * xi) + f(p_n * x_n) + f(p_{n+1} * x_{n+1}) <= (\sum_{i=1}^{n-1} pi * f(xi)) + (p_n * f(x_n))$
$+ (p_{n+1} * f(x_{n+1}))$

We know that $f(\sum_{i=1}^{n-1} pi * xi) <= \sum_{i=1}^{n-1} pi * f(xi)$ is true for any n >= 3, and so we just need to show that:

$$f(p_n * x_n) + f(p_{n+1} * x_{n+1}) <= (p_n * f(x_n)) + (p_{n+1} * f(x_{n+1}))$$

We can rewrite the equation above as:

$$f(p_nx_n + p_{n+1}x_{N=1}) <= p_n f(x_n) + p_{n+1}f(x_{n+1})$$

As we saw in the base case, this is the definition of a convex function and because we know that our function is convex, we can say that this is true.

For clarification of notation, let:
$f(\sum_{i=1}^{n-1} pi * xi) = A$, $f(p_n * x_n) + f(p_n * x_n) = B$
$(\sum_{i=1}^{n-1} pi * f(xi)) = C$, $(p_n * f(x_n)) + (p_{n+1} * f(x_{n+1})) = D$

We've shown that A <= C and B <= D, therefore we can conclude that A+B <= C+D, which we can sum up more formally to say that:

$$f(\sum_{i=1}^{n+1} pi * xi) <= \sum_{i=1}^{n+1} pi * f(xi),$$

which means our inductive case holds true.

Because the base case and inductive case both hold true, we can conclude that the whole proof holds true.

b. We want to show that:

(1) x^2 is convex, which we will prove directly, using the definition of a convex function.

Plugging in x^2 in as the function in the definition of a convex function gives:

$(tx_1 + (1-t)x_2)^2 <= tx_1^2 + (1-t)x_2^2$

➔ $(tx_1 + (1-t)x_2) * (tx_1 + (1-t)x_2) <= tx_1^2 + (1-t)x_2^2$

➔ $(tx_1)^2 + 2(tx_1(1-t)x_2) + ((1-t)x_2)^2 <= tx_1^2 + (1-t)x_2^2$

➔ $0 <= tx_1^2 + (1-t)x_2^2 – (tx_1)^2 - 2(tx_1(1-t)x_2) - ((1-t)x_2)^2$

➔ $0 <= t(1-t)(x_1-x_2)$

This expression is true and therefore x^2 is convex.

(2) For every $n \in N$, $(n \geq 2)$, and any $x_1, \ldots, x_n \in R$:

$$((1/n) \sum_{i=1}^{n} xi)^2 <= (1/n)\sum_{i=1}^{n} xi^2$$

We will prove this directly, using our conclusion from part a.

Let pi = 1/n, which will hold the condition that $p_1 + \ldots + p_n = 1$.

We can then plug pi = 1/n and $f(x) = x^2$ into the solution from part a, which gives:

$$(\sum_{i=1}^{n} (\frac{1}{n}) * xi)^2 <= \sum_{i=1}^{n} (\frac{1}{n}) * xi^2$$

We can then pull the (1/n) out front on both sides, leaving us with:

$$(1/n)(\sum_{i=1}^{n} xi)^2 <= (1/n)\sum_{i=1}^{n} xi^2$$

This is exactly the equation we want to prove and so we can conclude that the proof holds true.